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PRICE ADJUSTMENT MECHANISMS ENSURING THE STABILITY OF EQUILIBRIUM IN A MULTIDIMENSIONAL VERSION OF SCARF'S MODEL

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INTRODUCTORY NOTES

The studies of stability of a competitive equilibrium have a long-lasting history. An outstanding contribution to the problem was made by Herbert Scarf (Scarf, 1960). His famous counter-example demonstrating the possibility of global instability of the competitive equilibrium was initially formulated for the case of three economic agents and the Walrasian price adjustment mechanism (the tatonnement process). Recently Minagawa (2008) extended Scarf's treatment to a multi-dimensional case and a more general class of utility functions, however retaining an assumption on the Walrasian price dynamics. Kumar and Shubik (2004) developed Scarf's model in another direction and generalized the Walrasian price adjustment mechanism to the proportional-integral-derivative (PID) mechanism, however restricting their consideration to a three-dimensional case and numeric simulations only. In the current paper a synthesis of these two recent developments (Minagawa, 2008; Kumar, Shubik, 2004) is performed, and the stability of an N-dimensional Scarf's model with the PID price adjustment mechanism is studied. The consideration is essentially based on a previous work by the author (Kovalevsky, 2010) in which an important particular case of the PID mechanism, namely the proportional-integral (PI) mechanism, was treated in detail.

N-DIMENSIONAL SCARF'S MODEL. PRICE ADJUSTMENT MECHANISMS. EQUILIBRIA

Consider an exchange economy with N agents and N goods ($N \geq 3$). The agents and goods are enumerated by a subscript $k = 1, \ldots, N$ with an additional convention of cyclical closure (i. e. $k = N + 1$ is identical to $k = 1$, and $k = 0$ is identical to $k = N$). The agent k is assumed to be interested in goods k and $k + 1$ only, with the corresponding utility function

$$U_k(x_1, \ldots, x_k, x_{k+1}, \ldots, x_N) = \min(x_k, x_{k+1}), \quad (1)$$

where x_k is the amount of good k. The agent k is endowed by one unit of good k and by no other goods.

Let $P_k(t)$ be the price of good k. The Scarf's formula for the excess demand E_k (Scarf, 1960) then takes the form

$$E_k = \frac{P_{k-1}}{P_{k-1} + P_k} - \frac{P_{k+1}}{P_k + P_{k+1}} \quad (2)$$

(the dependence on time is omitted for brevity).

The Walrasian price adjustment mechanism (the classical tatonnement process) considered in (Scarf, 1960) is described by a dynamic system

$$\dot{\mathbf{P}} = k_0 \mathbf{E}, \quad (3)$$

where a dot above a variable denotes the time derivative, k_0 is a parameter determining the sensitivity of the market to changes in supply and demand, and N – dimensional vectors of prices and excess demands are introduced:

$$\mathbf{P} = (P_1, \ldots, P_N)^{\mathrm{T}}, \ \mathbf{E} = (E_1(\mathbf{P}), \ldots, E_N(\mathbf{P}))^{\mathrm{T}}, \quad (4)$$

(the superscript T denotes the transposition).

Kumar and Shubik (2004) proposed to generalize the Walrasian price adjustment mechanism to the PID form (the abbreviation is borrowed from the optimal control theory):

$$\dot{\mathbf{P}} = k_0 \mathbf{E} + k_1 \dot{\mathbf{E}} + k_2 \ddot{\mathbf{E}}, \quad (5)$$

where additional parameters k_1 and k_2 are introduced. In a particular case $k_2 = 0$ Eq. (5) takes the form

$$\dot{\mathbf{P}} = k_0 \mathbf{E} + k_1 \dot{\mathbf{E}} \quad (6)$$

and is called the PI price adjustment mechanism. This case was considered in detail in (Kovalevsky, 2010), and it was shown that the N-dimensional Scarf's economy, being globally unstable under the Walrasian price adjustment mechanism (Eq. (3)), becomes stable under the PI mechanism (Eq. (6)).

For all three price adjustment mechanisms considered above (Eqs. (4) – (6)) an equilibrium price vector $\mathbf{P}^*$ satisfies an equation $\mathbf{E} = 0$. Then it immediately follows from Eq. (2) that

$$P^*_{k-1} = P^*_{k+1}. \quad (7)$$

This means that in case of an *odd* number of goods all equilibrium prices are equal:

$$P_1^* = P_2^* = \ldots = P_N^* := P_C^*, \quad (8)$$

while in case of an *even* number of goods, all goods with even subscripts have the same price, and, analogously, the prices of goods with odd subscripts are also equal:

$$P_1^* = P_3^* = \ldots = P_{N-1}^* := P_A^*, \ P_2^* = P_4^* = \ldots = P_N^* := P_B^*. \quad (9)$$

STABILITY OF LINEARIZED SYSTEM

The stability of equilibria found in Sec. 2 in case of the PID price adjustment mechanism presented by Eq. (5) will be examined in linear approximation. Let $\mathbf{p}$ denote a small deviation from the equilibrium price vector: $\mathbf{p} = \mathbf{P} - \mathbf{P}^*$. After introducing the Jacobi matrix

$$\hat{\mathbf{D}} = \left.\frac{\partial \mathbf{E}}{\partial \mathbf{P}}\right|_{\mathbf{P}=\mathbf{P}^*}, \ D_{km} = \left.\frac{\partial E_k}{\partial P_m}\right|_{\mathbf{P}=\mathbf{P}^*} \quad (10)$$

Eq. (5) can be approximated by a linear dynamic system

$$\dot{\mathbf{p}} = \hat{\mathbf{D}}(k_0 \mathbf{p} + k_1 \dot{\mathbf{p}} + k_2 \ddot{\mathbf{p}}), \quad (11)$$

where the terms of the order $(\dot{p})^2$ have been neglected. Seeking a solution of Eq. (11) in the form

$$\mathbf{p}(t) = \mathbf{p}_0 e^{\lambda t}, \quad (12)$$

we find that the exponent must satisfy an equation

$$\det(g(\lambda)\hat{\mathbf{D}} - \lambda\hat{\mathbf{I}}) = 0, \quad \textbf{(13)}$$

where $\hat{\mathbf{I}}$ is a unit matrix, and a quadratic trinomial $g(\lambda)$ is introduced by the formula

$$g(\lambda) = k_2\lambda^2 + k_1\lambda + k_0. \quad \textbf{(14)}$$

In case of odd N the matrix $\hat{\mathbf{D}}$ appearing in Eq. (10) is a circulant of the form

$$\hat{\mathbf{D}} = \begin{pmatrix} 0 & -c & 0 & 0 & \ldots & 0 & c \\ c & 0 & -c & 0 & \ldots & 0 & 0 \\ 0 & c & 0 & -c & \ldots & 0 & 0 \\ & & & \ldots & & & \\ -c & 0 & 0 & 0 & \ldots & c & 0 \end{pmatrix}, \quad \textbf{(15)}$$

where

$$c = \frac{1}{4P_C^*}. \quad \textbf{(16)}$$

To solve Eq. (13) in this case, a theorem on eigenvalues of circulants can be applied. Some algebra analogous to calculations in (Kovalevsky, 2010, Appendix 1) leads to a quadratic equation for the exponent of the form

$$\lambda_k = -2ic\sin\frac{2\pi(k-1)}{N}g(\lambda_k),\ k = 1, \ldots, N. \quad \textbf{(17)}$$

Analogously, for even N the matrix $\hat{\mathbf{D}}$ takes the form

$$\hat{\mathbf{D}} = \begin{pmatrix} 0 & -a & 0 & 0 & 0 & \ldots & 0 & a \\ b & 0 & -b & 0 & 0 & \ldots & 0 & 0 \\ 0 & a & 0 & -a & 0 & \ldots & 0 & 0 \\ 0 & 0 & b & 0 & -b & \ldots & 0 & 0 \\ & & & \ldots & & & & \\ -b & 0 & 0 & 0 & 0 & \ldots & b & 0 \end{pmatrix}, \quad \textbf{(18)}$$

where

$$a = \frac{P_A^*}{(P_A^* + P_B^*)^2},\quad b = \frac{P_B^*}{(P_A^* + P_B^*)^2}. \quad \textbf{(19)}$$

Calculations analogous to those provided in (Kovalevsky, 2010, Appendix 2) yield in this case a quadratic equation

$$\lambda_k = -2i\sqrt{ab}\sin\frac{2\pi(k-1)}{N}g(\lambda_k),\ k = 1, \ldots, N. \quad \textbf{(20)}$$

It is a straightforward exercise in algebra of complex numbers to show that in case when all three parameters of the PID controller in Eq. (5), i. e. k_0, k_1 and k_2, are positive, the roots of Eqs. (17) and (20) have negative (in exceptional cases – zero) real components.

It follows from this fact that unlike the Walrasian price adjustment mechanism, the PID mechanism ensures the convergence of linearized Scarf's model to equilibrium for any number of agents $N \geq 3$.

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